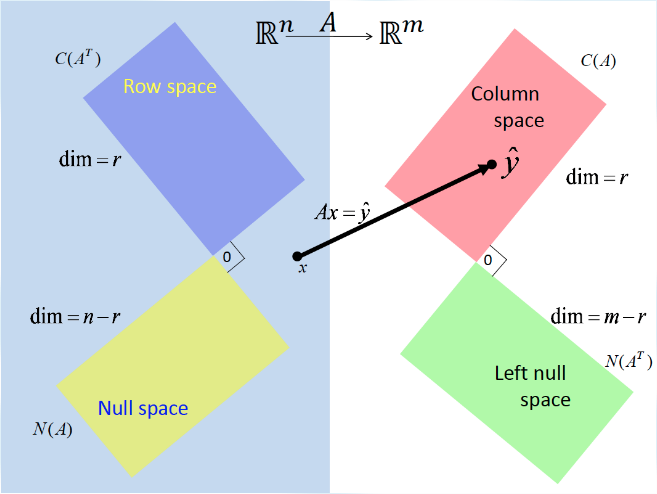
به نام او، به یاد او، برای او

Numerical Linear Algebra

Four Fundamental Subspaces of a Matrix

چهار زیرفضای اساسی یک ماتریس



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Four Fundamental Subspaces of a Matrix

❖ Column space of A:

$$C(A) = \{b = Ax : x \in \mathbb{R}^n\}$$

❖ Row space of $A =$ Column space of A^T:

$$C(A^T) = \{y = A^T z : z \in \mathbb{R}^m\}$$

❖ Null space of A:

$$N(A) = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}$$

❖ Left Null space of A:

$$N(A^T) = \{y \in \mathbb{R}^m : A^T y = \mathbf{0}\}$$

Linear Combination(review)

Definition: linear combination

Let $v_1, v_2, \ldots, v_k$ a set of vectors in a vector space $\mathbb{R}^n$. A *linear combination* of $v_1, v_2, \ldots, v_k$ is an expression of the form

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

Where $c_1, c_2, \ldots, c_k$ are scalars.

Definition: span

The *span* of $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$ is the set of *all linear combinations* of them

$$span\{v_1, v_2, \ldots, v_k\} = \{c_1v_1 + c_2v_2 + \cdots + c_kv_k \colon c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

Example:

What's the *span* of $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$?

$$span\{v_1, v_2\} = \{c_1v_1 + c_2v_2 \colon c_1, c_2 \in \mathbb{R}\}$$

$$= \left\{c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \colon c_1, c_2 \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix} c_1 + 2c_2 \\ c_1 - c_2 \end{bmatrix} \colon c_1, c_2 \in \mathbb{R}\right\}$$

$$= \mathbb{R}^2$$

Four Fundamental Subspaces of a Matrix

❖ *Column space of A*:

$$C(A) = \{b = Ax : x \in \mathbb{R}^n\}$$

How do you calculate the column space of a matrix?

1. Determine ***Row Echelon Form*** of A, $ref(A)$.
2. The ***pivot columns*** of $ref(A)$ form a ***basis*** for $C(A)$.
3. $C(A)$ = ***span*** of the ***pivot columns*** of $ref(A)$.

- The number of ***pivot columns*** is equal to the ***rank*** of A.
- The ***dimension*** of the column space is equal to the ***rank*** of A.

- ~~A has the same column space as $ref(A)$.~~

Four Fundamental Subspaces of a Matrix

❖ *Column space of A*:

$$C(A) = \{b = Ax : x \in \mathbb{R}^n\}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \xrightarrow[2+q\,1=0 \;\Rightarrow\; q=-2]{R_2 \leftarrow -2R_1 + R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 4 & 8 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 4 & 8 & 12 \end{bmatrix} \xrightarrow[4+q\,1=0 \;\Rightarrow\; q=-4]{R_3 \leftarrow -4R_1 + R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}2\\4\\0\end{matrix}} & \begin{matrix}3\\14\\0\end{matrix} \end{bmatrix}$$

pivot columns $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\}$

$$C(A) = \textit{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\} = \textit{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Four Fundamental Subspaces of a Matrix

- Row space of A = Column space of A^T:

$$C(A^T) = \{y = A^T z : z \in \mathbb{R}^m\}$$

How do you calculate the Row space of a matrix?

1. Determine ***Row Echelon Form*** of A, *ref*(A).
2. The ***nonzero rows*** of *ref*(A) form a ***basis*** for $C(A^T)$.
3. $C(A^T)$ = ***span*** of the ***nonzero rows*** of *ref*(A).

- The number of the ***nonzero rows*** of *ref*(A) is equal to the ***rank*** of A.
- The ***dimension*** of the ***row space*** is equal to the ***rank*** of A.
- A has the same ***row space*** as *ref*(A).
- ***Row space*** of A has the same *dimension* r and the same ***basis*** as *ref*(A).

Four Fundamental Subspaces of a Matrix

❖ Row space of A = Column space of A^T:

$$C(A^T) = \{y = A^T z : z \in \mathbb{R}^m\}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \xrightarrow[2+q\,1=0 \;\rightarrow\; q=-2]{R_2 \leftarrow -2R_1+R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 4 & 8 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 4 & 8 & 12 \end{bmatrix} \xrightarrow[4+q\,1=0 \;\rightarrow\; q=-4]{R_3 \leftarrow -4R_1+R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 0 & 0 & 0 \end{bmatrix} \qquad \textbf{nonzero rows}=\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 4 \\ 14 \end{bmatrix}^T \right\}$$

$$C(A^T) = \textit{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 4 \\ 14 \end{bmatrix}^T \right\}$$

Four Fundamental Subspaces of a Matrix

❖ Null space of A:

$$N(A) = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}$$

How do you calculate the Null space of a matrix?

1. Determine ***Row Echelon Form*** of A, *ref*(A).
2. Solve the system of equations *ref*(A)$\boldsymbol{x}$=$\mathbf{0}$.
3. $N(A) = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}$.

- The special solutions of $Ax = \mathbf{0}$ form a ***basis*** *for* $N(A)$.
- The ***dimension*** of the ***Null space*** is equal to n- r.
- A has the same ***null space*** as *ref*(A).
- A has the same *dimension* n-r and the same ***basis*** as *ref*(A).

Four Fundamental Subspaces of a Matrix

❖ Null space of A:

$$\mathrm{N}(A) = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \xRightarrow[2+q\,1=0 \;\rightarrow\; q=-2]{R_2 \leftarrow -2R_1+R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 4 & 8 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 4 & 8 & 12 \end{bmatrix} \xRightarrow[4+q\,1=0 \;\rightarrow\; q=-4]{R_3 \leftarrow -4R_1+R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 4x_3$$
$$x_2 = -3.5x_3$$

$$\mathrm{N}(A) = \textit{span}\left\{ t \begin{bmatrix} 4 \\ -3.5 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

Four Fundamental Subspaces of a Matrix

- Left Null space of A:

$$N(A^T) = \{y \in \mathbb{R}^m : A^T y = \mathbf{0}\}$$

How do you calculate the Left Null space of a matrix?

1. Determine ***Row Echelon Form*** of A^T, *ref*(A^T).
2. Solve the system of equations *ref*(A^T)***y***=**0**.
3. $N(A^T) = \{y \in \mathbb{R}^m : A^T y = \mathbf{0}\}$.

- The special solutions of $A^T y = \mathbf{0}$ form a ***basis*** *for* $N(A^T)$.
- The ***dimension*** of the ***Left Null space*** is equal to *m*- *r*.
- *A* has the same ***Left Null space*** as *ref*(A^T).

Four Fundamental Subspaces of a Matrix

- Left Null space of A:

$$N(A^T) = \{y \in \mathbb{R}^m : A^T y = \mathbf{0}\}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 8 & 8 \\ 3 & 20 & 12 \end{bmatrix} \qquad ref(A^T) = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_3 = ?$$
$$y_2 = 0$$
$$y_1 = -4y_3$$

$$\mathrm{N}(A^T) = \boldsymbol{span}\left\{ t \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

Four Fundamental Subspaces of a Matrix