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Scientific Computing

Systems of Linear Equations

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Linear Combination

Definition: linear combination

Let $v_1, v_2, \ldots, v_k$ a set of vectors in a vector space $\mathbb{R}^{\boldsymbol{n}}$. A *linear combination* of $v_1, v_2, \ldots, v_k$ is an expression of the form

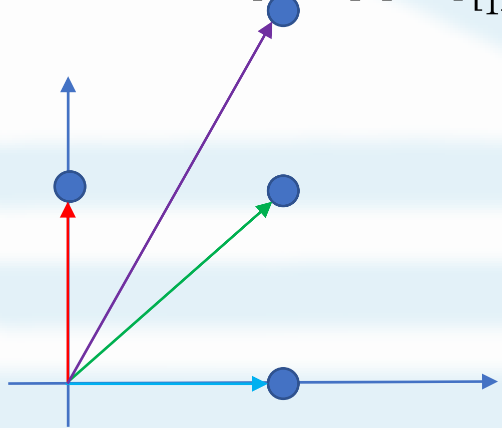
$$c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

Where $c_1, c_2, \ldots, c_k$ are scalars.

Question:

What's the *linear combination* of $v_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $v_2 = \begin{bmatrix}2\\-1\end{bmatrix}$ in $\mathbb{R}^2$?

$$c_1v_1 + c_2v_2 = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}c_1+2c_2\\c_1-c_2\end{bmatrix} \in \mathbb{R}^2, \quad c_1, c_2 \in \mathbb{R}$$



$$v_1 = \begin{bmatrix}1\\1\end{bmatrix}$$

$$v_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

$$v_3 = \begin{bmatrix}1\\0\end{bmatrix}$$

$$c_1 = 2, c_2 = 0, c_3 = -1$$

$$c_1v_1 + c_2v_2 + c_3v_3 =$$

$$2\begin{bmatrix}1\\1\end{bmatrix}+0\begin{bmatrix}0\\1\end{bmatrix} - 1\begin{bmatrix}1\\0\end{bmatrix}=$$

$$\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}-1\\0\end{bmatrix}=\begin{bmatrix}2+0-1\\2+0+0\end{bmatrix}$$

Span

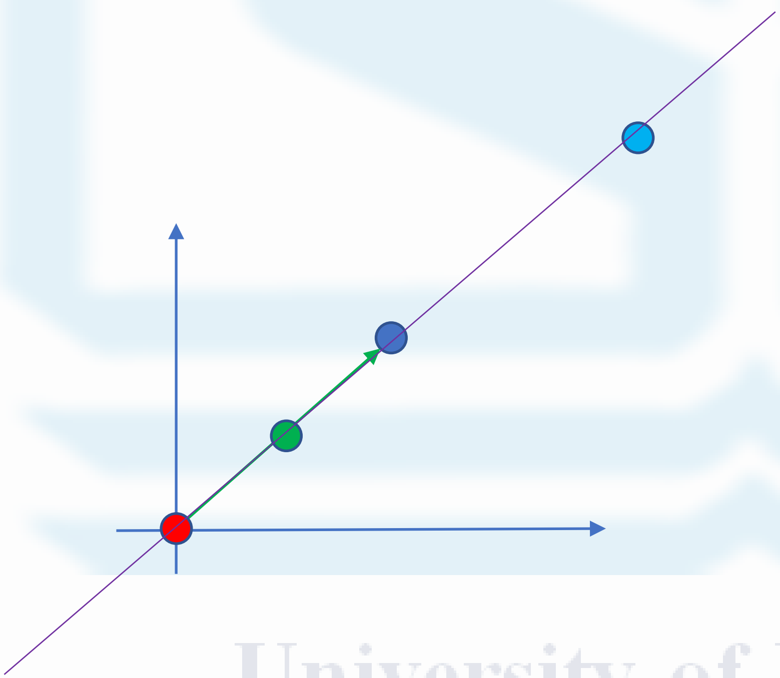
Definition: span

The *span* of $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$ is the set of *all linear combinations* of them

$$span\{v_1, v_2, \ldots, v_k\} = \{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \colon c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

Example 1: The span of a single, nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ is a line through the origin

$$span\{v\} = \{cv \in \mathbb{R}^n \colon c \in \mathbb{R}\}$$



$$c = 0 \quad ==> \quad cv = 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c = -0.5 \quad ==> \quad cv = 0.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}$$

$$c = 2 \quad ==> \quad cv = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\vdots$

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Span

Theorem:

Let $v_1, v_2, \ldots, v_k$ a set of vectors in a vector space $\mathbb{R}^{\boldsymbol{n}}$. The span of $v_1, v_2, \ldots, v_k$ is a subspace of $\mathbb{R}^{\boldsymbol{n}}$.

Question:

What's the span of $v_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $v_2 = \begin{bmatrix}2\\-1\end{bmatrix}$ in $\mathbb{R}^{\boldsymbol{2}}$?

$$
\begin{aligned}
span\{v_1, v_2\} &= \{c_1 v_1 + c_2 v_2 \colon c_1, c_2 \in \mathbb{R}\} \\
&= \left\{c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}2\\-1\end{bmatrix} \colon c_1, c_2 \in \mathbb{R}\right\} \\
&= \left\{\begin{bmatrix}c_1 + 2c_2\\c_1 - c_2\end{bmatrix} \colon c_1, c_2 \in \mathbb{R}\right\} \\
&= \mathbb{R}^{\boldsymbol{2}}
\end{aligned}
$$

Augmented Matrix

Definition: Augmented Matrix

If $\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{m \times n}}$ and $\boldsymbol{B} \in \mathbb{R}^{\boldsymbol{m \times p}}$, then the *augmented matrix* $[A|B] \in \mathbb{R}^{\boldsymbol{m \times n+p}}$ is the matrix $[A \;\; B]$, that is the matrix whose first n columns are the columns of A, and whose last p columns are the columns of B.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 \\ 6 & 5 \\ 3 & 4 \end{bmatrix}$$

$$\Rightarrow \quad [A|B] = \begin{bmatrix} 1 & 2 & 3 & 2 & 3 \\ 4 & 5 & 6 & 6 & 5 \\ 7 & 8 & 9 & 3 & 4 \end{bmatrix} \in \mathbb{R}^{\boldsymbol{3 \times 5}}$$

Reduced Row Echelon Form

Definition: Reduced Row Echelon Form(RREF)

A matrix $\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{m \times n}}$ is said to be *in reduced row echelon form* if, counting from the topmost row to the bottom-most,

1. Any row containing a *nonzero* entry precedes any row in which all the entries are *zero* (if any).
2. The *first nonzero* entry in each row is the only nonzero entry in its column.
3. The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

Example 1: The rduced row echelon form of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix}$ is:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix}$$

$$\boldsymbol{RREF}(A) = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3.5 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{\mathbf{3 \times 5}}$$

Example 2: The following matrices are not in reduced echelon form because they all fail some part of 3 (the first one also fails 2):

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

Example 3:

$$A = \begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ \boxed{2 \quad 8 \quad 20} \\ \boxed{4 \quad 8 \quad 12} \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \xRightarrow[2 + q\,1 = 0 \;\Rightarrow\; q = -2]{R_2 \leftarrow -2R_1 + R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 4 & 8 & 12 \end{bmatrix}$$

$$-2R_1 + R_2 = -2[1 \quad 2 \quad 3] + [2 \;\; 8 \;\; 20] = [0 \quad 4 \quad 14]$$

$$\xRightarrow[4 + q\,1 = 0 \;\Rightarrow\; q = -4]{R_3 \leftarrow -4R_1 + R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-4R_1 + R_3 = -4[1 \quad 2 \quad 3] + [4 \;\; 8 \;\; 12] = [0 \quad 0 \quad 0]$$

Reduced Row Echelon Form

Example 3(cont.):

$2 + q\,4 = 0 \Rightarrow q = -\frac{1}{2}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 14 \\ 0 & 0 & 0 \end{bmatrix} \xRightarrow{R_1 \leftarrow -\frac{1}{2}R_2 + R_1} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 4 & 14 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xRightarrow{R_2 \leftarrow \frac{1}{4}R_2} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3.5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \Rightarrow rref(A) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3.5 \\ 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

Example 4:

$$A = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

$2 + q\,6 = 0 \rightarrow q = -\frac{2}{6} = -\frac{1}{3}$

$$A = \begin{bmatrix} 6 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \xrightarrow{R_2 \leftarrow -\frac{1}{3}R_1 + R_2} \begin{bmatrix} 6 & 2 & 3 \\ 0 & -\frac{22}{3} & 19 \\ 4 & 8 & 12 \end{bmatrix}$$

$$-\frac{1}{3}R_1 + R_2 = -\frac{1}{3}[6 \;\; 2 \;\; 3] + [2 \;\; 8 \;\; 20] = [0 \;\; -\frac{22}{3} \;\; +19]$$

$4 + q\,6 = 0 \rightarrow q = \frac{-2}{3}$

$$\xrightarrow{R_3 \leftarrow \frac{-2}{3}R_1 + R_3} \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ 0 & 4 & 14 \\ 0 & -\frac{20}{3} & 10 \end{bmatrix}$$

$$-4R_1 + R_3 = \frac{-2}{3}[6 \;\; 2 \;\; 3] + [4 \;\; 8 \;\; 12] = [0 \; -\frac{20}{3} \;\; 10]$$

Reduced Row Echelon Form

Example 4(cont.):

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ 0 & 4 & 14 \\ 0 & -\frac{20}{3} & 10 \end{bmatrix} \xrightarrow{R_2 \leftarrow \frac{1}{4}R_2} \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ 0 & 1 & \frac{14}{4} \\ 0 & -\frac{20}{3} & 10 \end{bmatrix}$$

$\frac{1}{3} + q\ 1 = 0 \rightarrow q = -\frac{1}{3}$

$$\xrightarrow{R_1 \leftarrow -\frac{1}{3}R_2 + R_1} \begin{bmatrix} 1 & 0 & \frac{-11}{12} \\ 0 & 1 & \frac{14}{4} \\ 0 & -\frac{20}{3} & 10 \end{bmatrix}$$

$-\frac{20}{3} + q\ 1 = 0 \rightarrow q = \frac{20}{3}$

$$\xrightarrow{R_3 \leftarrow +\frac{20}{3}R_2 + R_3} \begin{bmatrix} 1 & 0 & \frac{-11}{12} \\ 0 & 1 & \frac{14}{4} \\ 0 & 0 & \frac{100}{3} \end{bmatrix}$$

$+\frac{20}{3}R_2 + R_3 = \frac{-2}{3}[6\ \ 2\ \ 3] + [4\ \ 8\ \ 12] = [0\ \ -\frac{20}{3}\ \ \ 10]$

Reduced Row Echelon Form

Example 4(cont.):

$$\begin{bmatrix} 1 & 0 & \frac{-11}{12} \\ 0 & 1 & \frac{14}{4} \\ 0 & 0 & \frac{100}{3} \end{bmatrix} \xrightarrow{R_3 \leftarrow \frac{1}{\frac{100}{3}} R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Linearly dependent

Definition: linearly dependent

A set of vectors $v_1, v_2, ..., v_k$ from vector space $\mathbb{R}^n$ is said to be ***linearly dependent*** if at least one of the vectors in the set can be defined as a linear combination of the others ; or if there exist scalers c_1, $c_2, ..., c_k$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \mathbf{0}.$$

Example.

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad t = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \quad \Longrightarrow \quad t = v + 2v$$

Definition: linearly independent

A set of vectors $v_1, v_2, ..., v_k$ from vector space $\mathbb{R}^n$ is said to be ***linearly independent,*** if no vector in the set can be written in this way, then the vectors are said to be *linearly independent*; or if linear combination of, $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \mathbf{0}$, results that all scalers $c_1, c_2, ...$ and c_k are zero.

Example.

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}: \qquad if \quad c_1 v_1 + c_2 v_2 = \mathbf{0} \quad \Longrightarrow \quad c_1 = c_2 = \mathbf{0}$$

Rank of a Matrix

Definition: rank of a matrix

The ***rank of a matrix*** is defined as

(a) the maximum number of linearly independent column vectors in the matrix or

(b) the maximum number of linearly independent row vectors in the matrix.

Both definitions are equivalent.

Definition: rank of a matrix

The ***rank of a matrix*** is defined as the number of leading 1s in rref(A).

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 20 \\ 4 & 8 & 12 \end{bmatrix} \Rightarrow rref(A) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3.5 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of a Matrix

Theorem: *If* $A \in \mathbb{R}^{m\times n}$, $P \in \mathbb{R}^{m\times m}$ *and* $Q \in \mathbb{R}^{n\times n}$, P *and* Q *invertible, then*

(a) rank(AQ)=rank(A),
(b) rank(PA)=rank(A),
(c) rank(PAQ)=rank(A).

Corollary: *Elementary row and column operations on a matrix are rank-preserving.*

Systems of Linear Equations

A system of m **linear equations in** n **unknowns** is a set of m equations, numbered from 1 to m going down, each in n variables x_i which are multiplied by real coefficients a_{ij}, whose sum equals some real number b_j:

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 & + & \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 & + & \cdots + a_{2n}x_n & = b_2 \\ \vdots \qquad\quad \vdots & & \ddots \qquad \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 & + & \cdots + a_{mn}x_n & = b_m \end{array}$$

Matrix form of A system of m **linear equations in** n **unknowns:**

$$Ax = b \qquad \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b}$$

Systems of Linear Equations

$$
\begin{array}{ccccccc}
a_{11}x_1 + a_{12}x_2 & + & \cdots + a_{1n}x_n & = b_1 \\
a_{21}x_1 + a_{22}x_2 & + & \cdots + a_{2n}x_n & = b_2 \\
\vdots \qquad \vdots & & \ddots \qquad \vdots & \vdots \\
a_{m1}x_1 + a_{m2}x_2 & + & \cdots + a_{mn}x_n & = b_m
\end{array}
$$

Remarks:

- A *system* of equations is considered ***overdetermined*** if $m>n$, there are more equations than unknowns.
- A *system* of equations is considered ***underdetermined*** if $m<n$, there are fewer equations than unknowns.
- If $\boldsymbol{b} = \mathbf{0}$, the system is said to be ***homogeneous***, while if $\boldsymbol{b} \neq \mathbf{0}$ it is said to be ***nonhomogeneous***.
- Every nonhomogeneous system $A\boldsymbol{x}=\boldsymbol{b}$ has an associated or corresponding homogeneous system $A\boldsymbol{x}=\mathbf{0}$.
- Each system $A\boldsymbol{x}=\boldsymbol{b}$, homogeneous or not, has an associated or corresponding *augmented matrix* is the $[A|\boldsymbol{b}] \in \mathbb{R}^{m\times(n+1)}$.

Solution of Systems of Linear Equations, *Ax*=*b*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \ddots \quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Remarks:

- A linear system is ***consistent*** if *it has solution*. The equation *A***x**=***b*** is consistent if the matrix *A* has a pivot position in every row.

$$A = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 1 & 3.5 \\ 0 & 0 & -7 \end{bmatrix}$$

- A linear system is ***consistent*** *iff* $\mathrm{rank}(A) = rank([A|\boldsymbol{b}])$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 6 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} 6 \\ 15 \\ 21 \end{bmatrix} \Rightarrow [A|\boldsymbol{b}] = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 15 \\ 7 & 8 & 6 & 21 \end{bmatrix} \Rightarrow \mathrm{rank}(A) = rank([A|\boldsymbol{b}]) = 3$$

Solution of Systems of Linear Equations, *Ax*=*b*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Remarks:

- A linear system is ***inconsistent*** if it has no solution, and otherwise it is said to be ***consistent***. The equation *A***x**=***b*** is consistent if the matrix *A* has a pivot position in every row.

$$A = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 1 & 3.5 \\ 0 & 0 & 0 \end{bmatrix}$$

- When the system is ***inconsistent***, it is possible to derive a ***contradiction*** from the equations, that may always be rewritten as the statement 0 = 1.

$$\begin{aligned} 2x_1 + 3x_2 &= 6 \\ 2x_1 + 3x_2 &= 10 \end{aligned}$$

- An *overdetermined system* is *almost always* ***inconsistent*** (it has no solution) when constructed with random coefficients.

Solution of Systems of Linear Equations, *Ax=b*

$$\begin{array}{cccccc}
a_{11}x_1 + a_{12}x_2 & + & \cdots + a_{1n}x_n & = b_1 \\
a_{21}x_1 + a_{22}x_2 & + & \cdots + a_{2n}x_n & = b_2 \\
\vdots \qquad\quad \vdots & & \ddots \qquad \vdots & \vdots \\
a_{m1}x_1 + a_{m2}x_2 & + & \cdots + a_{mn}x_n & = b_m
\end{array}$$

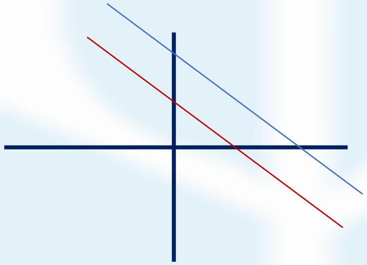
Remarks:

- The solution set of $A\boldsymbol{x}=\boldsymbol{b}$ is denoted here by K . A system is either consistent , by which we mean $K \neq \emptyset$, or inconsistent, by which we mean $K = \emptyset$.
- Two systems of linear equations are called equivalent if they have the same solution set.
- For example the systems $A\boldsymbol{x}=\boldsymbol{b}$ and $B\boldsymbol{x}=\boldsymbol{c}$, where $[B|\boldsymbol{c}]=$ rref($[A|\ \boldsymbol{b}]$) are equivalent.

Three possible outcomes for general *Ax=b* $(A \in \mathbb{R}^{m\times n})$

❖ *Ax*=***b*** is inconsistent and has ***no*** solution

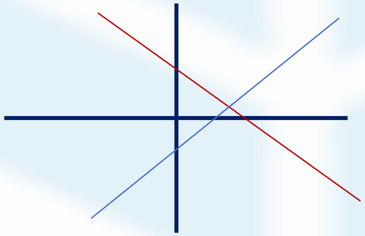
$$2x_1 + 3x_2 = 6$$
$$4x_1 + 6x_2 = 10$$



parallel but not identical lines

❖ *Ax*=***b*** is consistent and has an ***unique*** solution.

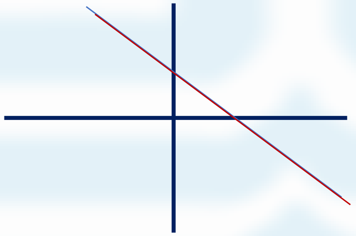
$$2x_1 + 3x_2 = 6$$
$$-1x_1 + 1x_2 = -1$$



non-parallel lines

❖ *Ax*=***b*** is consistent and has ***infinite*** solution.

$$2x_1 + 3x_2 = 12$$
$$4x_1 + 6x_2 = 24$$



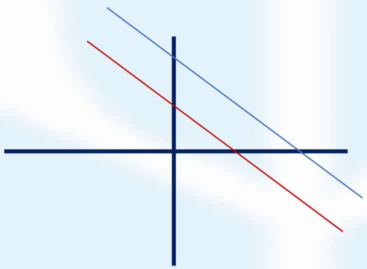
identical lines

Three possible outcomes for *Ax*=*b* ($A \in \mathbb{R}^{n\times n}$)

- If $\det(A) = 0$ and *Ax*=***b*** is inconsistent, , then *Ax=b* has ***no*** solution.

$2x_1 + 3x_2 = 6$

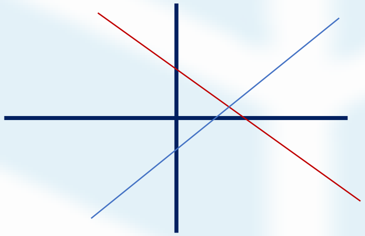
$4x_1 + 6x_2 = 10$



- If $\det(A) \neq 0$ and *Ax*=***b*** is consistent, then *Ax=b* and has an ***unique*** solution.

$2x_1 + 3x_2 = 6$

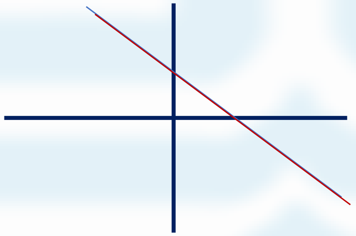
$-1x_1 + 1x_2 = -1$



- If $\det(A) = 0$ and *Ax*=***b*** is consistent, then *Ax=b* and has ***infinite*** solution.

$2x_1 + 3x_2 = 12$

$4x_1 + 6x_2 = 24$



Affine Space

Definition. Nullspace of *A*:

$$\mathrm{N}(A) = \{x \in \mathbb{R}^n | Ax=\mathbf{0}\}.$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \mathrm{N}(A) = \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 = -2x_2\right\}$$

$$= \left\{t\begin{bmatrix} -2 \\ 1 \end{bmatrix} : t \in \mathbb{R}\right\}$$

Theorem: *The solution set K of any system A***x**=**b** *of m linear equations in n unknowns is an* ***affine space***, *namely*

$$K = s + \mathrm{N}(A),$$

where, N(*A*) *is nullspace of A and s is a particular solution s of A***x**=**b**.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \Rightarrow \mathrm{N}(A) = \left\{t\begin{bmatrix} -2 \\ 1 \end{bmatrix} : t \in \mathbb{R}\right\}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \Rightarrow s := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow K = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left\{t\begin{bmatrix} -2 \\ 1 \end{bmatrix} : t \in \mathbb{R}\right\}$$

Solution of Systems of Linear Equations, *Ax=b*

Theorem: *Let A$\mathbf{x}$=$\mathbf{b}$ be a system of m linear equations in n unknowns. The system has exactly one solution $A^{-1}\mathbf{b}$ iff A is invertible.*

Corollary : *If A$\mathbf{x}$=$\mathbf{b}$ is a system of m linear equations in n unknowns and and it's augmented matrix [A|$\mathbf{b}$] is transformed into a reduced row echelon matrix $[A'|\mathbf{b}']$ by a finite sequence of elementary row operations, then*

(1) A$\mathbf{x}$=$\mathbf{b}$ is inconsistent iff $rank(A') \neq rank([A'|\mathbf{b}'])$ iff $[A'|\mathbf{b}']$ contains a row in which the only nonzero entry lies in the last column, the $\mathbf{b}'$ column.

(2) A$\mathbf{x}$=$\mathbf{b}$ is consistent iff $[A'|\mathbf{b}']$ contains no row in which the only nonzero entry lies in the last column, the $\mathbf{b}'$ column.

inconsistent

$$A=\begin{bmatrix}1&2&3\\2&8&20\\4&8&12\end{bmatrix}\quad \boldsymbol{b}=\begin{bmatrix}6\\15\\21\end{bmatrix} \Rightarrow [A|\boldsymbol{b}]=\begin{bmatrix}1&2&3&6\\2&8&20&15\\4&8&12&21\end{bmatrix} \Rightarrow [A'|\boldsymbol{b}']=\begin{bmatrix}1&0&-4&0\\0&1&3.5&0\\0&0&0&1\end{bmatrix}$$

$$\Rightarrow 2=\mathrm{rank}(A')\neq \mathrm{rank}([A'|\mathbf{b}'])=3$$

consistent

$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&6\end{bmatrix}\quad \boldsymbol{b}=\begin{bmatrix}6\\15\\21\end{bmatrix} \Rightarrow [A|\boldsymbol{b}]=\begin{bmatrix}1&2&3&6\\4&5&6&15\\7&8&6&21\end{bmatrix} \Rightarrow [A'|\boldsymbol{b}']=\begin{bmatrix}1&0&0&*\\0&1&0&*\\0&0&1&1.4\end{bmatrix}$$

$$\Rightarrow \mathrm{rank}(A')=\mathrm{rank}([A'|\mathbf{b}'])=3$$

Solution of Systems of Linear Equations, *Ax=b*

Theorem: *Let* $A\mathbf{x}=\mathbf{b}$ *be a system of m linear equations in n unknowns. If* $B \in \mathbb{R}^{m\times m}$ *is invertible, then the system* $(BA)\mathbf{x}=B\mathbf{b}$ *is equivalent to* $A\mathbf{x}=\mathbf{b}$.

Proof: *If* K *is the solution set of* $A\mathbf{x}=\mathbf{b}$ *and* K' *is the solution set for* $(BA)\mathbf{x}=B\mathbf{b}$*, then*

$$\begin{aligned} \boldsymbol{w} \in K \quad &\Leftrightarrow \quad A\boldsymbol{w}=\boldsymbol{b} = (B^{-1}B)\boldsymbol{b} \\ &\Leftrightarrow \quad (BA)\boldsymbol{w} = B\boldsymbol{b} \\ &\Leftrightarrow \boldsymbol{w} \in K' \end{aligned}$$

so $K = K'$.

Gaussian Elimination

Corollary: *If* $A\boldsymbol{x}=\boldsymbol{b}$ *is a system of* m *linear equations in* n *unknowns, then* $A'x = \boldsymbol{b}'$ *is equivalent to* $A\boldsymbol{x}=\boldsymbol{b}$ *if* $[A'|\boldsymbol{b}']$ *is obtained from* $[A|\boldsymbol{b}]$ *by a finite number of elementary row operations.*

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